C19

Differentiation and the gradient function

CHAPTER ASSIGNMENT · 377 MARKS

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given. Show your working: write the gradient function first, then substitute. A calculator may be used for checking. This whole chapter is Extended content.

Practice questions from *Cambridge IGCSE Mathematics Core and Extended* (4th edition), Chapter 19 (Differentiation and the gradient function), for classroom use. Graph descriptions and value tables stand in for the book's figures. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 19.1 · The gradient of a curve

19.1.1 (a) Using the proof above as a guide, find the gradient of the function $f(x) = x^2$ when: (i) $x = 2$ · (ii) $x = 3$ · (iii) $x = -1$. [6]

(b) Make a table of the values of x and the corresponding values of the gradient of the function f(x). [2]

(c) Looking at the pattern in your results, complete the sentence below. For the function $f(x) = x^2$, the gradient is ... [1]

19.1.2 (a) Find the gradient of the function $f(x) = 2x^2$ when: (i) $x = 1$ · (ii) $x = 2$ · (iii) $x = -2$. [6]

(b) Looking at the pattern in your results, complete the sentence below. For the function $f(x) = 2x^2$, the gradient is ... [1]

19.1.3 (a) Find the gradient of the function $f(x) = \frac{1}{2}x^2$ when: (i) $x = 1$ · (ii) $x = 2$ · (iii) $x = 3$. [6]

(b) Looking at the pattern in your results, complete the sentence below. For the function $f(x) = \frac{1}{2}x^2$, the gradient is ... [1]

Exercise 19.2 · The gradient function from first principles

19.2.1 Find, from first principles, the gradient function of each function. Use the worked example above as a guide. (The worked example finds the gradient function of $f(x) = x^2 + x$ as $2x + 1$.) **[18]**

(a) $f(x) = x^3$ · (b) $f(x) = 3x^2$ · (c) $f(x) = x^2 + 2x$

(d) $f(x) = x^2 - 2$ · (e) $f(x) = 3x - 3$ · (f) $f(x) = 2x^2 - x + 1$

19.2.2 Copy and complete the table below using your gradient functions from the previous question and Exercise 19.1 Q1–3.

Function f(x)	Gradient function f′(x)
x^2	
$2x^2$	
$\frac{1}{2}x^2$	
$x^2 + x$	$2x + 1$
x^3	
$3x^2$	
$x^2 + 2x$	
$x^2 - 2$	
$3x - 3$	
$2x^2 - x + 1$	

19.2.3 Look at your completed table for Q2. Describe any patterns you notice between a function and its gradient function. [3]

Exercise 19.3 · The rule for ax^n

19.3.1 Calculate the gradient function of each of the following functions.

$f(x) = x^4$	$f(x) = x$	$f(x) = 3x^2$	$f(x) = 5x^3$	$f(x) = 6x^3$	$f(x) = 8x^7$

19.3.2 Calculate the gradient function of each of the following functions.

$f(x) = \frac{1}{3}x^3$	$f(x) = \frac{1}{4}x^4$	$f(x) = \frac{1}{4}x^2$	$f(x) = \frac{1}{2}x^4$	$f(x) = \frac{2}{5}x^3$	$f(x) = (2/9)x^3$

Exercise 19.4 · Differentiation

19.4.1 Differentiate each expression with respect to x.

a	b	c	d	e	f	g	h	i
$5x^3$	$7x^2$	$4x^6$	$\frac{1}{4}x^2$	$\frac{2}{3}x^6$	$\frac{3}{4}x^5$	5	6x	1/8

19.4.2 Differentiate each expression with respect to x. [18]

(a) $3x^2 + 4x$ · (b) $5x^3 - 2x^2$ · (c) $10x^3 - \frac{1}{2}x^2$

(d) $6x^3 - 3x^2 + x$ · (e) $12x^4 - 2x^2 + 5$ · (f) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 4$

(g) $-3x^4 + 4x^2 - 1$ · (h) $-6x^5 + 3x^4 - x + 1$ · (i) $-\frac{3}{4}x^6 + \frac{2}{3}x^3 - 8$

19.4.3 Differentiate each expression with respect to x. [18]

(a) $(x^3 + x^2) \div x$ · (b) $(4x^3 - x^2) \div x^2$ · (c) $(6x^3 + 2x^2) \div 2x$

(d) $(x^3 + 2x^2) \div 4x$ · (e) $3x(x + 1)$ · (f) $2x^2(x - 2)$

(g) $(x + 5)^2$ · (h) $(2x - 1)(x + 4)$ · (i) $(x^2 + x)(x - 3)$

Exercise 19.5 · Using other variables

19.5.1 Differentiate each of the following with respect to t: (a) $y = 3t^2 + t$ · (b) $v = 2t^3 - t^2$ · (c) $m = 5t^3 - t^2$ [6]

19.5.2 Calculate the derivative of each of the following functions: (a) $y = x(x + 4)$ · (b) $r = t(1 - t)$ · (c) $v = t(1/t + t^2)$ · (d) $p = r^2(2/r - 3)$ [8]

19.5.3 Differentiate each of the following with respect to t: (a) $y = (t + 1)(t - 1)$ · (b) $r = (t - 1)(2t + 2)$ · (c) $v = (2t^2/3 + 1)(t - 1)$ [6]

Exercise 19.6 · The second derivative

19.6.1 Find d^2y/dx^2 for each of the following.

$y = 2x^3$	$y = x^4 - \frac{1}{2}x^2$	$y = \frac{1}{3}x^6$	$y = 3x^2 - 2$	$y = x^2/4$	$y = 3x$

19.6.2 Find the second derivative of each function: (a) $v = x^2(x - 3)$ · (b) $P = \frac{1}{2}x^2(x^2 + x)$ · (c) $t = x^{-1}(x + x^3)$ · (d) $y = (x^2 + 1)(x^3 - x)$ [8]

Exercise 19.7 · Gradient of a curve at a point

19.7.1 Find the gradient of each function at the given value of x. [7]

(a) $f(x) = x^2$; $x = 3$ · (b) $f(x) = \frac{1}{2}x^2 - 2$; $x = -3$ · (c) $f(x) = 3x^3 - 4x^2 - 2$; $x = 0$

(d) $f(x) = -x^2 + 2x - 1$; $x = 1$ · (e) $f(x) = -\frac{1}{2}x^3 + x - 3$; $x = -1$, $x = 2$ · (f) $f(x) = 6x$; $x = 5$

19.7.2 The number of newly infected people, N, on day t of a stomach bug outbreak is given by $N = 5t^2 - \frac{1}{2}t^3$.

(a) Calculate the number of new infections N when: (i) $t = 1$ · (ii) $t = 3$ · (iii) $t = 6$ · (iv) $t = 10$. [4]

(b) Calculate the rate of new infections with respect to t, i.e. calculate dN/dt. [2]

(c) Calculate the rate of new infections at the following times: (i) $t = 1$ · (ii) $t = 3$ · (iii) $t = 6$ · (iv) $t = 10$. [4]

(d) Plot a graph of the equation $N = 5t^2 - \frac{1}{2}t^3$ for the values of t in the range $0 \leq t \leq 10$. [4]

(e) Explain your answers to part (a), using your graph to support your explanation. [2]

(f) Explain your answers to part (c), using your graph to support your explanation. [2]

19.7.4 A weather balloon is released from the ground. Its height in metres, h, after time in hours t, is given by the formula $h = 30t^2 - t^3$ when $t \leq 20$.

(a) Calculate the height of the balloon when: (i) $t = 3$ · (ii) $t = 10$. [2]

(b) Calculate the rate at which the balloon is climbing with respect to time t. [2]

(c) Calculate the rate of ascent when: (i) $t = 2$ · (ii) $t = 5$ · (iii) $t = 20$. [3]

(d) The graph of h against t is shown in the book. (It rises from h = 0 at t = 0 to h = 4000 at t = 20, steepest in the middle and flattening off at t = 20.) Values from the graph: [3]

t (hours)	0	2	4	6	8	10	12	14	16	18	20
h (m)	0	112	416	864	1408	2000	2592	3136	3584	3888	4000

Referring to your graph, explain your answers to part (c).

(e) Estimate from your graph the time when the balloon was climbing at its fastest rate. Explain your answer. [2]

Exercise 19.8 · Finding x when the gradient is given

19.8.1 Find the coordinate of the point P on each of the following curves, at the given gradient. [18]

(a) $f(x) = x^2 - 3$, gradient = 6 · (b) $f(x) = 3x^2 + 1$, gradient = 15 · (c) $f(x) = 2x^2 - x + 4$, gradient = 7

(d) $f(x) = \frac{1}{2}x^2 - 3x - 1$, gradient = −3 · (e) $f(x) = \frac{1}{3}x^2 + 4x$, gradient = 6 · (f) $f(x) = -\frac{1}{5}x^2 + 2x + 1$, gradient = 4

19.8.2 Find the coordinate(s) of the point(s) on each of the following curves, at the given gradient. [16]

(a) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x$, gradient = 6 · (b) $f(x) = \frac{1}{3}x^3 + 2x^2 + 6x$, gradient = 3

(c) $f(x) = \frac{1}{3}x^3 - 2x^2$, gradient = −4 · (d) $f(x) = x^3 - x^2 + 4x$, gradient = 5

19.8.3 A stone is thrown vertically downwards off a tall cliff. The distance (s) it travels in metres is given by the formula $s = 4t + 5t^2$, where t is the time in seconds after the stone's release.

(a) What is the rate of change of distance with time, ds/dt? (This represents the velocity.) [2]

(b) How many seconds after its release is the stone travelling at a velocity of 9 m/s? [2]

(c) The speed of the stone as it hits the ground is 34 m/s. How many seconds after its release did the stone hit the ground? [2]

(d) Using your answer to part (c), calculate the distance the stone falls and hence the height of the cliff. [2]

19.8.4 The temperature inside a pressure cooker (T) in degrees Celsius is given by the formula $T = 20 + 12t^2 - t^3$, where t is the time in minutes after the cooking started and $t \leq 8$.

(a) Calculate the initial (starting) temperature of the pressure cooker. [1]

(b) What is the rate of temperature increase with time? [2]

(c) What is the rate of temperature increase when: (i) $t = 1$ · (ii) $t = 4$ · (iii) $t = 8$. [3]

(d) The pressure cooker was switched off when $dT/dt = 36$. How long after the start could the pressure cooker have been switched off? Give both possible answers. [4]

(e) What was the final temperature of the pressure cooker if it was switched off at the greater of the two times calculated in part (d). [2]

Exercise 19.9 · Equation of the tangent at a given point

19.9.1 For the function $f(x) = x^2 - 3x + 1$

(a) Calculate the gradient function. [1]

(b) Calculate the gradient of the curve at the point A(2, 1). [1]

(c) A tangent is drawn to the curve at A. What is the gradient of the tangent? [1]

(d) Calculate the equation of the tangent. Give your answer in the form $y = mx + c$. [2]

19.9.2 For the function $f(x) = 2x^2 - 4x - 2$

(a) Calculate the gradient of the curve at $x = 2$. [2]

(b) A tangent is drawn to the curve at the point $(2, -2)$. Calculate the equation of the tangent. Give your answer in the form $y = mx + c$. [2]

19.9.3 A tangent is drawn to the curve $f(x) = \frac{1}{2}x^2 - 4x - 2$ at the point P(0, −2).

(a) Calculate the gradient of the tangent at P. [2]

(b) Calculate the equation of the tangent. Give your answer in the form $y = mx + c$. [2]

19.9.4 A tangent, T1, is drawn to the curve $f(x) = -x^2 + 4x + 1$ at point A(4, 1).

(a) Calculate the gradient of the tangent at A. [2]

(b) Calculate the equation of the tangent. Give your answer in the form $y = mx + c$. [2]

(c) Another tangent to the curve, T2, is drawn at point B(2, 5). Calculate the equation of T2. [2]

19.9.5 A tangent, T1, is drawn to the curve $f(x) = -\frac{1}{4}x^2 - 3x + 1$ at point P(−2, 6).

(a) Calculate the equation of T1. [3]

(b) Another tangent to the curve, T2, with equation $y = 10$, is drawn at point Q. Calculate the coordinates of point Q. [3]

(c) T1 and T2 are extended so that they intersect. Calculate the coordinates of their point of intersection. [2]

19.9.6 The equation of a tangent T, drawn to the curve $f(x) = -\frac{1}{2}x^2 - x - 4$ at P, has equation $y = -3x - 2$.

(a) Calculate the gradient function of the curve. [1]

(b) What is the gradient of the tangent T? [1]

(c) What are the coordinates of point P? [3]

Exercise 19.10 · Stationary points

19.10.1 For each function, calculate: (i) the gradient function · (ii) the coordinates of any stationary points. [12]

(a) $f(x) = x^2 - 6x + 13$ · (b) $f(x) = x^2 + 12x + 35$ · (c) $f(x) = -x^2 + 8x - 13$ · (d) $f(x) = -6x + 7$

19.10.2 For each function, calculate: (i) the gradient function · (ii) the coordinates of any stationary points. [16]

(a) $f(x) = x^3 - 12x^2 + 48x - 58$ · (b) $f(x) = x^3 - 12x$

(c) $f(x) = x^3 - 3x^2 - 45x + 8$ · (d) $f(x) = \frac{1}{3}x^3 + (3/2)x^2 - 4x - 5$

19.10.3 For questions 3 to 6: (a) Calculate the gradient function. (b) Calculate the coordinates of any stationary points. (c) Determine the nature of each stationary point. (d) Calculate the value of the y-intercept. (e) Sketch the graph of the function. Q3: $f(x) = 1 - 4x - x^2$ [8]

19.10.4 $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 3$ (parts (a) to (e) as above) [10]

19.10.5 $f(x) = -\frac{2}{3}x^3 + 3x^2 - 4x$ (parts (a) to (e) as above) [10]

19.10.6 $f(x) = x^3 - (9/2)x^2 - 30x + 4$ (parts (a) to (e) as above) [10]

Student assessment 1

SA1.1 Find the gradient function of the following: (a) $y = x^3$ · (b) $y = 2x^2 - x$ · (c) $y = -\frac{1}{2}x^2 + 2x$ · (d) $y = \frac{2}{3}x^3 + 4x^2 - x$ [6]

SA1.2 Differentiate the following functions with respect to x: (a) $f(x) = x(x + 2)$ · (b) $f(x) = (x + 2)(x - 3)$ · (c) $f(x) = (x^3 - x) \div x$ · (d) $f(x) = (x^3 + 2x^2) \div 2x$ [8]

SA1.3 Find the second derivative of the following functions: (a) $y = x^4 - 3x^2$ · (b) $s = 2t^5 - t^3$ [4]

SA1.4 Find the gradient of the following curves at the given values of x: (a) $f(x) = ½x^2 + x$; $x = 1$ · (b) $f(x) = -x^3 + 2x^2 + x$; $x = 0$ · (c) $f(x) = (x - 3)(x + 8)$; $x = ¼$ [6]

SA1.5 A stone is dropped from the top of a cliff. The distance (s) it falls is given by the equation $s = 5t^2$, where s is the distance in metres and t the time in seconds.

(a) Calculate the velocity v, by differentiating the distance s with respect to time t. [2]

(b) Calculate the stone's velocity after 3 seconds. [1]

(c) The stone hits the ground travelling at 42 m s^{-1}, calculate: (i) how long it took for the stone to hit the ground · (ii) the height of the cliff. [4]

Student assessment 2

SA2.1 The function $f(x) = x^3 + x^2 - 1$ has a gradient of zero at points P and Q, where the x coordinate of P is less than that of Q.

(a) Calculate the gradient function f′(x). [1]

(b) Calculate the coordinates of P. [3]

(c) Calculate the coordinates of Q. [2]

(d) Determine which of the points P or Q is a maximum. Explain your method clearly. [3]

SA2.2 (a) Explain why the point A(1, 1) lies on the curve $y = x^3 - x^2 + x$ [1]

(b) Calculate the gradient of the curve at A. [2]

(c) Calculate the equation of the tangent to the curve at A. [2]

(d) Calculate the equation of the normal to the curve at A. [3]

SA2.3 For the function $f(x) = (x - 2)^2 + 3$

(a) Calculate $f'(x)$. [2]

(b) Determine the coordinates of the stationary point. [2]

SA2.4 For the function $f(x) = x^4 - 2x^2$

(a) Calculate $f'(x)$. [1]

(b) Determine the coordinates of any stationary points. [4]

(c) Determine the nature of any stationary point. [3]

(d) Find where the graph intersects or touches: (i) the y-axis · (ii) the x-axis [3]

(e) Sketch the graph of f(x). [3]

Check yourself

1. Explain, using a sketch, why the gradient of the chord PQ gets closer to the gradient of the tangent at P as Q moves towards P. [2]

2. A student says the derivative of $(2x^3 + x^2) \div x$ is $(6x^2 + 2x) \div 1$. Explain the mistake and give the correct derivative. [3]

3. Differentiate $f(x) = (x - 1)(x^2 + 2)$ and find the gradient of the curve at $x = 2$. [3]

4 The curve $y = x^2 + bx + 5$ has a stationary point at $x = 3$. Find b, and say whether the point is a maximum or a minimum. [3]

5 Find the equation of the tangent to $y = x^3 - 2x$ at the point where $x = 1$. [3]

Chapter quiz

Q1 P is (1, 1) on $y = x^2$. Q moves along the curve towards P. What does the gradient of PQ get closer to?

A	B	C	D
1	2	0	It keeps growing

Q2 Differentiate $5x^4$.

A	B	C	D
$20x^3$	$5x^3$	$20x^5$	$4x^3$

Q3 What is the derivative of the constant 7?

A	B	C	D
7	7x	1	0

Q4 Find dy/dx for $y = x^3 - 4x + 1$.

A	B	C	D
$3x^2 - 4x$	$3x^2 - 4$	$x^2 - 4$	$3x^2 - 4 + 1$

Q5 Differentiate $(2x^3 + x^2) \div x$.

A	B	C	D
$6x^2 + 2x$	$4x + 1$	$2x^2 + x$	$6x + 2$

Q6 A stone has $s = 10t - 4.9t^2$. What is its velocity ds/dt?

A	B	C	D
$10 - 9.8t$	$10 - 4.9t$	-9.8	$10t - 9.8$

Q7 Find d^2y/dx^2 for $y = x^3 - 2x^2$.

A	B	C	D
$3x^2 - 4x$	$6x - 4$	$6x$	$3x - 4$

Q8 What is the gradient of $y = x^3 + x - 6$ at $x = -1$?

A	B	C	D
-2	2	4	-6

Q9 At which x does $y = x^2 - 2x + 1$ have gradient 4?

A	B	C	D
$x = 1$	$x = 2$	$x = 3$	$x = 4$

Q10 What is the tangent to $y = x^2$ at the point (3, 9)?

A	B	C	D
$y = 6x - 9$	$y = 6x + 9$	$y = 3x$	$y = 2x + 3$

Q11 Where is the stationary point of $y = x^2 - 4x + 7$?

A	B	C	D
(4, 7)	(2, 3)	(−2, 19)	(2, 7)

Q12 At a stationary point, $d^2y/dx^2 = -4$. What kind of point is it?

A	B	C	D
A minimum	A maximum	Not a stationary point	A y-intercept

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/o-level-mathematics/chapter-19-differentiation-and-the-gradient-function/